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TORSION IN THE MAPPING CLASS GROUP AND ITS COHOMOLOGY

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Let S_g denote a closed orientable surface of genus g. Recall that Γ_g, the mapping class group of S_g, is defined to be the group of path components of the group of orientation preserving homeomorphisms of S_g. Baer and Nielsen showed that Γ_g is isomorphic to the group $\mathrm{Out}^+(\pi_1 S_g)$ of orientation preserving outer automorphisms of the fundamental group of S_g. Observe that $\Gamma_1 \cong \mathrm{Out}^+(\mathbb{Z}\oplus\mathbb{Z}) \cong \mathrm{Sl}_2(\mathbb{Z})$ is an arithemetic group. In fact all the mapping class groups are closely related to arithmetic groups and they enjoy many properties arithmetic groups do. In particular, Γ_g has a torsion-free subgroup of finite index and finite cohomological dimension (i.e., $\mathrm{vcd}(\Gamma_g)<\infty$). This implies that in large dimensions with respect to g, $H^*(\Gamma_g;\mathbb{Z})$ has p-torsion if and only if p is a torsion prime for Γ_g. In addition, Γ_g is a virtual duality group [8] and therefore $H^i(\Gamma_g;\mathbb{Z})$ is finitely generated for all i, and finite for $i>\mathrm{vcd}(\Gamma_g)$. Another remarkable fact is Harer's stability theorem [9] which states that $H^k(\Gamma_g;\mathbb{Z})$ is independent of g for $g>3k$. We put as a notational device

$$H^k(\Gamma)=H^k(\Gamma_g;\mathbb{Z}) \quad \text{for } g\gg k.$$

In this paper we will construct in a systematic way torsion classes in $H^*(\Gamma)$, using the Chern classes of the canonical representation $\Gamma_g \to \mathrm{Gl}_{2g}(\mathbb{Z})$, induced by the action of Γ_g on $H_1(S_g;\mathbb{Z})$. Some homological torsion for the mapping class group has previously been constructed by Charney and Lee in [2], where they consider general characteristic classes for certain moduli spaces.

The following theorem was announced in [7].

Main Theorem. *The stable cohomology group $H^{4k}(\Gamma)$ contains an element of order $E_{2k}=\mathrm{den.}(B_{2k}/2k)$, the denominator of $B_{2k}/2k$, where B_{2k} denotes the $2k$-th Bernoulli number ($B_2=1/6$, $B_4=1/30$, $B_6=1/42,\dots,$ and $E_2=12$, $E_4=120$, $E_6=252,\dots$).*

The numbers E_{2k}, which in [4] are denoted by $E_{\mathbb{Q}}(2k)$, grow very rapidly with k and involve eventually any prime power infinitely often. This may be seen from the well known formula [4]

$$E_{2k} = \operatorname{lcm}\{n \mid 2k \equiv 0 \bmod \phi(n)\}$$

where ϕ denotes the Euler function. Our torsion elements in $H^{4k}(\Gamma)$ are, up to a factor two, of the same order as the universal Chern classes c_{2k} in $H^{4k}(\mathrm{Gl}(\mathbb{Z}); \mathbb{Z})$ or $H^{4k}(\mathrm{Sp}(\mathbb{Z}); \mathbb{Z})$, and they are therefore best possible in an obvious sense [5].

In order to prove the Main Theorem we need results concerning the Chern classes of representations of cyclic groups and also about the existence of torsion in Γ_g. We get the latter by studying suitable $\mathbb{Z}/n\mathbb{Z}$-actions on S_g and observing that these actions eventually 'stabilize' in the sense that they always exist for g large with respect to n. This discussion also leads to unstable torsion in $H^*(\Gamma_g; \mathbb{Z})$.

The remainder of this paper is organized as follows. In Section 1 we review results about Chern classes of representations of cyclic groups and prove certain special properties of these needed later on. In Section 2, we discuss the construction of torsion in Γ_g by exhibiting effective orientable actions of cyclic groups on S_g, using techniques of branched covering spaces. In Section 3, we use the results of Sections 1 and 2 to obtain the torsion in $H^*(\Gamma_g; \mathbb{Z})$ which is related to the cyclic subgroups of Γ_g by means of Chern classes of the canonical representation $\Gamma_g \to \mathrm{Gl}_{2g}(\mathbb{Z})$. The use of cyclic subgroups of Γ_g rather than arbitrary finite subgroups has no influence on the type of results we prove, as one can see from general properties of Chern classes of integral representations [4]. There is an obvious relationship between torsion in Γ_g and torsion in $H^*(\Gamma_g; \mathbb{Z})$ in the 'unstable range' (i.e., $*$ large compared to g) and we construct an upper bound for the order of torsion classes in $H^k(\Gamma_g; \mathbb{Z})$ for $k > \mathrm{vcd}(\Gamma_g)$. In Section 4, we make use of certain cyclic subgroups of Γ_g which are embedded in a way such that the methods developed in the earlier sections provide torsion in the stable range of $H^*(\Gamma_g; \mathbb{Z})$. Our general Theorem 4.4 is independent of Harer's stability result; it implies the Main Theorem using the fact that $H^k(\Gamma_g; \mathbb{Z})$ is independent of g for g large compared with k. By applying Harer's precise stability range, we can find for every prime $p \geq 17$ a group Γ_g which has no element of order p but for which $H^*(\Gamma_g; \mathbb{Z})$ does have p-torsion. This exceptional p-torsion lies of course in degrees below $\mathrm{vcd}(\Gamma_g)$; it is related to p-torsion in some Γ_{g-s}, $s > 0$, via cohomological stability.

1. Chern classes of representations of cyclic groups

The basic references for this section are [3] and [4]. Recall that for a complex representation $\varphi: G \to \mathrm{Gl}_k(\mathbb{C})$ of the discrete group G the Chern classes $c_i(\varphi) \in H^{2i}(G; \mathbb{Z})$ are defined as Chern classes of the flat $\mathbb{C}^k$-bundle over $K(G, 1)$ classified by $B\varphi: K(G, 1) \to B\mathrm{Gl}_k(\mathbb{C})$.

If $A \subset \mathbb{C}$ is a subring and $\varphi: G \to \mathrm{Gl}_k(A)$ a representation over A, we will write

$c_i(\varphi)$ for the i-th Chern class of the associated complex representation $G \to \mathrm{Gl}_k(A) \to \mathrm{Gl}_k(\mathbb{C})$.

Let now $G = \mathbb{Z}/n\mathbb{Z}$. It is well known that over $\mathbb{Q}$ the group $\mathbb{Z}/n\mathbb{Z}$ has a unique faithful irreducible representation, which we will denote by

$$\sigma_n : \mathbb{Z}/n\mathbb{Z} \to \mathrm{Gl}_{\phi(n)}(\mathbb{Q})$$

The degree of σ_n is $\phi(n)$, ϕ the Euler function; σ_n can be realized by the obvious action of $\mathbb{Z}/n\mathbb{Z}$ on the cyclotomic field $\mathbb{Q}(\zeta_n)$, where ζ_n denotes a primitive n-th root of 1, and we refer thus to σ_n as the *cyclotomic* representation of $\mathbb{Z}/n\mathbb{Z}$. In case $n = p$ a prime number, σ_p is equivalent to the reduced regular representation $\mathbb{Q}[\mathbb{Z}/p\mathbb{Z}]/\mathbb{Q}$, and has character

$$\chi_{\sigma_p}(x) = \begin{cases} -1, & \text{if } x \neq 0, \\ p-1, & \text{if } x = 0 \end{cases}$$

where $x \in \mathbb{Z}/p\mathbb{Z}$. For $\alpha > 1$, σ_{p^α} is equivalent to the representation induced from the reduced regular representation of $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{Z}/p^\alpha\mathbb{Z}$; this follows, for instance, from the fact that there is only one faithful representation of $\mathbb{Z}/p^\alpha\mathbb{Z}$ of degree $p^{\alpha-1}(p-1)$ over $\mathbb{Q}$, cf. [3].

From the well-known formula for the character of an induced representation (see e.g. [14]), we infer

$$\chi_{\sigma_{p^\alpha}}(x) = \begin{cases} 0, & \text{if } px \neq 0, \\ -p^{\alpha-1}, & \text{if } px = 0 \text{ and } x \neq 0, \\ p^{\alpha-1}(p-1), & \text{if } x = 0 \end{cases} \tag{1.1}$$

where $x \in \mathbb{Z}/p^\alpha\mathbb{Z}$.

If we consider σ_n as a complex representation, it decomposes into a sum of 1-dimensional representations, $\sigma_n \sim \varrho_1 + \varrho_2 + \cdots + \varrho_{\phi(n)}$, the sum being taken over all faithful $\mathbb{C}$-irreducible representations of $\mathbb{Z}/n\mathbb{Z}$. Observe that $c_1(\varrho_j)$ is a generator of $H^2(\mathbb{Z}/n\mathbb{Z}; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. The Cartan formula shows therefore that

$$c_{\phi(n)}(\sigma_n) = \prod_j c_1(\varrho_j) \in H^{2\phi(n)}(\mathbb{Z}/n\mathbb{Z}; \mathbb{Z})$$

has (maximal) order n.

Let us write $k\varphi$ for the k-fold direct sum of a representation ϱ (thus $\chi_{k\varphi} = k\chi_\varphi$). We will need the following property of σ_{p^α}.

Proposition 1.2. *Let $\varrho : \mathbb{Z}/p^\alpha\mathbb{Z} \to \mathrm{Gl}_k(\mathbb{Q})$ be a $\mathbb{Q}$-representation. Suppose that in the decomposition of ϱ into $\mathbb{Q}$-irreducible representations σ_{p^α} occurs with multiplicity m, where m is not divisible by p. Then, for every $j > 0$,*

$$(c_{\phi(p^\alpha)}(\varrho))^j \in H^{2j\phi(p^\alpha)}(\mathbb{Z}/p^\alpha\mathbb{Z}; \mathbb{Z})$$

has order p^α.

Proof. If we decompose ϱ into a sum of 1-dimensional complex representations, we see that every one-dimensional faithful representation ϱ_j ($j=1,\dots,p^{\alpha-1}(p-1)$) of $\mathbb{Z}/p^\alpha$ occurs with multiplicity m in ϱ. The total Chern class $c(\varrho)=1+c_1(\varrho)+c_2(\varrho)+\cdots$ thus satisfies

$$c(\varrho)\equiv c(\textstyle\sum m\varrho_j)=c(\sum \varrho_j)^m \mod pH^*(\mathbb{Z}/p^\alpha;\mathbb{Z})$$

and, since ϱ is defined over $\mathbb{Q}$, we know from [3] that $c_k(\varrho)$ has order strictly smaller than p^α for $1\le k<p^{\alpha-1}(p-1)$.

It follows that

$$c(\varrho)\equiv 1+mc_{\phi(p^\alpha)}(\textstyle\sum \varrho_j)+\text{higher terms} \mod pH^*(\mathbb{Z}/p^\alpha;\mathbb{Z}).$$

We have already observed that $c_{\phi(p^\alpha)}(\sum \varrho_j)$ has order p^α. Since m is prime to p, the assertion follows hence from the well-known ring structure of $H^*(\mathbb{Z}/p^\alpha\mathbb{Z};\mathbb{Z})$.

2. Torsion in the mapping class group

Let $x\in\Gamma_g$ be of finite order n. If $g=1$, one has $\Gamma_g\cong \mathrm{Sl}_2(\mathbb{Z})$ and therefore $n\le 6$; if $g>1$ it is well known that $n\le 84(g-1)$ (cf. [6]). The element x can be represented by an orientation preserving homeomorphism $S_g\to S_g$ with isolated fixed points and period n; by abuse of notation we shall also write x for this diffeomorphism. The quotient map $\pi: S_g\to S_g/\langle x\rangle$ is a branched covering of closed orientable surfaces, with branched covering transformation group $\langle x\rangle\cong\mathbb{Z}/n\mathbb{Z}$. We call $r\in S_g$ a *ramification point*, if $|\pi^{-1}(\pi(r))|<n$; the image $\pi(r)\in S_{\bar g}$ of a ramification point r is called a *branch point*, and the integer $n/|\pi^{-1}(\pi(r))|$ is called the order of the branch point $\pi(r)$. According to the Riemann–Hurwitz equation, one has

$$2g-2=n(2\bar g-2)+n\left(\sum_{i=1}^{b}(1-1/n_i)\right) \tag{2.1}$$

if $S_g\to S_g/\langle x\rangle=S_{\bar g}$ is a branched covering with group $\mathbb{Z}/n\mathbb{Z}$ and b branch points of order $(n_1,\dots,n_b)$. The existence of torsion elements in Γ_g is therefore equivalent to the existence of suitable branched covering spaces $S_g\to S_{\bar g}$. The precise relationship is as follows (see also Harvey [10]).

Proposition 2.2. *Let $n\ge 2$ and $b\ge 0$ and suppose given divisors $n_1,\dots,n_b$ of n such that $n_i>1$ for $1\le i\le b$. Then, for every $\bar g\ge 0$ there exists a (regular) branched covering of orientable surfaces $S_g\to S_{\bar g}$ with (branched) covering transformation group $\mathbb{Z}/n\mathbb{Z}$ and b branch points of orders $n_1,\dots,n_b$ respectively if and only if*

(i) *for every prime power p^s dividing some n_i there exists an index $j\ne i$ such that p^s divides n_j,*

(ii) *in case $m=\operatorname{lcm}\{n_i\mid 1\le i\le b\}$ is even, the number of branch points with m/n_i odd is even,*

(iii) *in case $\bar g=0$, every prime power dividing n has to divide some n_i.*

Proof. We first show that the conditions (i)-(iii) are necessary. Let $\pi: S_g \to S_{\bar{g}}$ be a branched covering with (possible empty) set of branch points $\{y_1, \dots, y_b\} \subset S_{\bar{g}}$ and group $\mathbb{Z}/n\mathbb{Z}$. Then

$$S_g - \pi^{-1}\{y_1, \dots, y_b\} \to S_{\bar{g}} - \{y_1, \dots, y_b\}$$

is a regular covering, inducing a short exact sequence

$$\pi_1(S_g - \pi^{-1}\{y_1, \dots, y_b\}) \to \pi_1(S_{\bar{g}} - \{y_1, \dots, y_b\}) \xrightarrow{\partial} \mathbb{Z}/n\mathbb{Z}.$$

We can choose a presentation of $\pi_1(S_{\bar{g}} - \{y_1, \dots, y_b\})$ of the form

$$\left\langle a_1, b_1, \dots, a_{\bar{g}}, b_{\bar{g}}, x_1, \dots, x_b \,\middle|\, \prod_{i=1}^{\bar{g}} [a_i, b_i] x_1 \cdots x_b = 1 \right\rangle$$

with $|\partial(x_i)| = n_i$, $1 \le i \le b$. Since $\mathrm{im}(\partial)$ is abelian, $-\partial(x_i) = \partial(x_1 \cdots \hat{x}_i \cdots x_b)$ and $|\partial(x_i)| = \mathrm{lcm}\{|\partial x_j|, j \ne i\}$, which implies (i). Let $m = \mathrm{lcm}\{n_i \mid 1 \le i \le b\}$. Then $\partial(x_i) \in \mathbb{Z}/m\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z}$. Let us now assume m even and let $s: \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be the unique surjective map. Clearly, $s\partial(x_i) = 0$ implies $2n_i \mid m$ and therefore $s\partial(x_1 \cdots x_b) = 0$ implies (ii). The surjectivity of ∂ implies (iii). Conversely, we construct the desired branched covering as follows. First we consider the case of a prime power $n = p^\alpha > 1$ and we put $m = p^\beta = \mathrm{lcm}\{n_i \mid 1 \le i \le b\}$; in case $m = 1$ we have $b = 0$ and $S_g \to S_{\bar{g}}$ is the regular covering space associated with the kernel of a surjection $\pi_1(S_{\bar{g}}) \to \mathbb{Z}/n\mathbb{Z}$ (note that $\bar{g} > 0$ in this case because of (iii)). We will now assume $m = p^\beta > 1$. As before, we write

$$\pi_1(S_{\bar{g}} - \{y_1, \dots, y_b\}) = \langle a_1, b_1, \dots, a_{\bar{g}}, b_{\bar{g}}, x_1, \dots, x_b \mid \pi[a_i, b_i] x_1 \cdots x_b = 1 \rangle$$

and we will construct a surjective homomorphism $\partial: \pi_1(S_{\bar{g}} - \{y_1, \dots, y_b\}) \to \mathbb{Z}/n\mathbb{Z}$ such that $\partial(x_i)$ has order n_i for $1 \le i \le b$.

Case p odd: According to (i) there are (at least) two indices i, j with $n_i = n_j = p^\beta$, say $n_1 = n_2 = p^\beta$. For $3 \le i \le b$ we put $\partial(x_i) = w_i \in \mathbb{Z}/p^\beta \subset \mathbb{Z}/p^\alpha$ where $|w_i| = n_i$. If $w_3 + \cdots + w_b = w$ has order $p^\gamma, \gamma < \beta$, we put $\partial x_1 = w_1$, $|w_1| = p^\beta$ and $\partial x_2 = -(\partial x_1 + w)$. If $\gamma = \beta$, we put $\partial(x_1) = w$ and $\partial(x_2) = -2w$ (note that $|-2w| = p^\beta$ since p is odd). Then $\partial(x_1 \cdots x_b) = 0$ and we can extend ∂ to a surjective homomorphism $\pi_1(S_{\bar{g}} - \{y_1, \dots, y_b\}) \to \mathbb{Z}/n\mathbb{Z}$ (in case $\bar{g} = 0$, one makes use of (iii)). The kernel of ∂ defines a regular covering $S_g - \{z_1, \dots, z_r\} \to S_{\bar{g}} - \{y_1, \dots, y_b\}$ and, by compactification, we obtain a branched covering $S_g \to S_{\bar{g}}$ with g determined by the Riemann-Hurwitz formula.

Case $p = 2$: As before, we assume $n_1 = n_2 = 2^\beta$ and we set $\partial(x_i) = w_i$ for $i \le 3 \le b$, where $w_i \in \mathbb{Z}/2 \subset \mathbb{Z}/2^\alpha$ has order n_i. Condition (ii) implies that $w = w_3 + w_4 + \cdots + w_b$ has order strictly smaller than 2^β, since an even number of w_i's have (maximal) order 2^β. Let $v \in \mathbb{Z}/2^\beta$ be a generator. Then $-w - v \in \mathbb{Z}/2^\beta$ is also a generator and we set $\partial(x_1) = v$, $\partial(x_2) = -w - v$, which implies $\partial(x_1, \dots, x_b) = 0$. Thus ∂ can be extended to a surjective homomorphism $\pi_1 \to \mathbb{Z}/n$ and as before we obtain the desired branched covering $S_g \to S_{\bar{g}}$.

In the case of general n, we proceed as follows. We define $\partial(x_i) \in \mathbb{Z}/n\mathbb{Z} = \prod \mathbb{Z}/p^{\alpha(p)}$ by prescribing its p-primary components $\partial(x_i)_p \in \mathbb{Z}/p^{\alpha(p)}$. This is done by only using the p-primary parts $n_i(p)$ of the n_i's and then proceeding as before (if $n_i(p)=1$ we put, of course, $\partial(x_i)_p = 0$).

By means of (2.2) we obtain self maps $S_g \to S_g$, given by (branched) covering transformations. Such maps do always represent non-trivial elements in Γ_g, if $g>1$, as one can see by the following classical result.

Lemma 2.3. *let $S_g \to S_{\bar{g}}$ be a (branched) covering with $g>1$ and let $T: S_g \to S_g$ be a covering transformation of period n. Then T represents an element of order n in Γ_g.*

Proof. We choose a complex structure in $S_{\bar{g}}$ and we equip S_g with the induced complex structure such that $S_g \to S_{\bar{g}}$ is a holomorphic map. Then $T: S_g \to S_g$ is a holomorphic automorphism. The whole group of holomorphic automorphisms $\mathrm{Aut}(S_g)$ is finite for $g>1$ and it is a classical result [6] that $\mathrm{Aut}(S_g)$ is faithfully represented in $H_1(S_g; \mathbb{Z})$. Hence $\mathrm{Aut}(S_g)$ maps injectively to Γ_g and the result follows.

3. Unstable torsion

We will construct in this section torsion classes in $H^*(\Gamma_g; \mathbb{Z})$ related to cyclic subgroups $\mathbb{Z}/n\mathbb{Z} \subset \Gamma_g$. The term *unstable* is used here to indicate that we are interested in choosing g small compared with n; the resulting torsion in $H^*(\Gamma_g; \mathbb{Z})$ is then in general outside the stable range as determined by Harer [9]. We define

$$g(n) = \min\{g>0 \mid \mathbb{Z}/n\mathbb{Z} \subset \Gamma_g\}.$$

Since $\Gamma_1 \cong \mathrm{Sl}_2(\mathbb{Z})$, we have $g(2)=g(3)=g(4)=g(6)=1$, and $g(n)>1$ for $n=5$ or $n>6$. A general formula for $g(n)$ may be found in [10]. It is easy to compute $g(n)$ for any particular n, using (2.2) and (2.3), in conjunction with the Riemann-Hurwitz equation associated to the cyclic branched covering $S_{g(n)} \to S_{\bar{g}}$:

$$2g(n) - 2 = n(2\bar{g} - 2) + n \sum_{i=1}^{b} (1 - 1/n_i). \tag{3.1}$$

One has to choose $\bar{g} \geq 0$, $b \geq 0$ and divisors $n_1, \ldots, n_b$ of n in accordance to (2.2) in a way that the right hand side of (3.1) assumes a minimal value >1 (we may restrict to the cases $n=5$ or $n>6$ for which $g(n)>1$).

For $n=p^\alpha$ a prime power, one finds the following simple formula for $g(n)$.

Lemma 3.2. *Let p be a prime and $\alpha \geq 1$ an integer. Suppose that $p^\alpha > 2$. Then*

$$g(p^\alpha) = \tfrac{1}{2}\phi(p^\alpha)$$

where ϕ denotes Euler's function.

Proof. We have already observed that $g(3)=g(4)=1$. It suffices therefore to consider the case $p^\alpha>4$ and thus $g(p^\alpha)>1$. The equation (3.1) then assumes the form

$$g(p^\alpha)=p^\alpha(\bar{g}-1)+\tfrac{1}{2}\left(bp^\alpha-\sum_{i=1}^{b}p^{\alpha-\alpha_i}\right)+1$$

where, in case $\bar{g}=0$, $\alpha_i=\alpha$ for at least two indices. The right hand side therefore assumes a minimal value >1 for $\bar{g}=0$, $b=3$, $\alpha_1=\alpha_2=\alpha$, $\alpha_3=1$ (note that $\alpha>1$ in case $p=2$ and thus 2.2(ii) is satisfied). We obtain hence $2g(p^\alpha)=p^\alpha-p^{\alpha-1}=\phi(p^\alpha)$.

It will also be useful to know which of the Γ_g's contain elements of order p. The only torsion primes which occur in $\Gamma_1\cong \mathrm{Sl}_2(\mathbb{Z})$ are 2 and 3. For $g>1$ the following holds.

Lemma 3.3. *Let $g>1$. Then Γ_g contains an element of prime order p if and only if g is of the form*

$$g=up+v(p-1)/2$$

where $(u,v)\in\mathbb{Z}\times\mathbb{Z}$, $u\geq 0$, $v\geq -2$ and $v\neq -1$.

Proof. If $g>1$, then Γ_g contains an element of order p if and only if there exists a cyclic (branched) covering $S_g\to S_{\bar{g}}$ of order p. For such a covering (2.1) reduces to $2g-2=p(2\bar{g}-2)+pb-b$, or, $g=\bar{g}p+(b-2)(p-1)/2$. We put now $u=\bar{g}$ and $v=b-2$, hence $u\geq 0$, $v\geq -2$. Since $b=1$ is impossible by 2.2(i), we must have $v\neq -1$; the integrality of g ensures that b is even in case $p=2$, in accordance with 2.2(ii). The equation $g=up+v(p-1)/2$ with $g>1$ as above permits us therefore to construct a covering transformation $T: S_g\to S_g$ which, by (2.3), represents an element of order p in Γ_g.

It is obvious from (3.3) that although $\mathbb{Z}/n\mathbb{Z}\subset\Gamma_{g(n)}$, in general $\mathbb{Z}/n\mathbb{Z}\not\subset\Gamma_{g(n)+k}$ for $k>0$. However, as well see in the next section $\mathbb{Z}/n\mathbb{Z}\subset\Gamma_{g(n)+k}$ for k large. There are therefore only finitely many 'gaps' with $\mathbb{Z}/n\mathbb{Z}\not\subset\Gamma_{g(n)+k}$.

Table 1 is a complete list of gaps for small values of n. It is an exercise, using (2.2) and (2.3).

The following large gap index will be used in Section 4.

Corollary 3.4. *Suppose p is a prime ≥ 5 and let $\kappa(p)=p(p-3)/2-(p-1)/2$. Then $\Gamma_{\kappa(p)}$ contains no p-torsion.*

Proof. Suppose $\Gamma_{\kappa(p)}$ has p-torsion. Then $\kappa(p)=up+v(p-1)/2$ for some $u\geq 0$, $v\geq -2$, $v\neq -1$ according to (3.3). Reducing mod $(p-1)/2$ this implies $\kappa(p)\equiv u$. On the other hand

Table 1

n	$g(n)$	$g>g(n)$ with $\mathbb{Z}/n\mathbb{Z}\not\subset\Gamma_g$
≤ 4	1	none
5	2	3
6	1	none
7	3	4, 5, 11
8	2	none
9	3	5
10	2	3
11	5	6, 7, 8, 9, 13, 14, 17, 18, 19, 24, 28, 29, 39
12	3	none
13	6	7, 8, 9, 10, 11, 15, 16, 17, 20, 21, 22, 23, 28, 29, 34, 35, 41, 46, 47, 59
14	3	4, 5, 11
15	4	none
16	4	5, 9
17	8	9, 10, 11, 12, 13, 14, 15, 19, 20, 21, 22, 23, 26, 27, 28, 29, 30, 31, 36, 37, 38, 39, 43, 44, 45, 46, 47, 53, 54, 55, 60, 61, 62, 63, 70, 71, 77, 78, 79, 87, 94, 95, 111
18	4	5

$$\kappa(p)=p(p-3)/2-(p-1)/2\equiv -1 \bmod (p-1)/2$$

and therefore $u=s(p-1)/2-1$ for some $s>0$; $s>1$ would produce a value for $\kappa(p)$ which is too big, since $v\geq -2$. Hence $u=(p-3)/2$, which implies $v=-1$, a value which is not admissible. Therefore $\Gamma_{\kappa(p)}$ contains no p-torsion.

There is an obvious relationship between torsion in Γ_g and torsion in $H^*(\Gamma_g;\mathbb{Z})$. This stems from the classical fact that Γ_g contains a torsion-free subgroup of finite index and finite cohomological dimension ($=\mathrm{vcd}(\Gamma_g)$). Of course, $\mathrm{vcd}(\Gamma_1)=1$ since $\mathrm{Sl}_2(\mathbb{Z})$ contains a free subgroup of finite index; for $g>1$ it is known that $\mathrm{vcd}(\Gamma_g)=4g-5$ (cf. [8]). The following is a well known property of virtually torsion free groups of finite virtual cohomological dimension (cf. [1]).

Lemma 3.5. *Let p be a prime and $g\geq 1$. Then Γ_g contains an element of order p if and only if there exists a $k>\mathrm{vcd}(\Gamma_g)$ such that $H^k(\Gamma_g;\mathbb{Z})$ contains an element of order p.*

The torsion in $H^k(\Gamma_g;\mathbb{Z})$ in the range $k>\mathrm{vcd}(\Gamma_g)\geq 4g-5$ lies above the stable range (which is the range $g>3k$). One can give a bound for this 'unstable torsion' as follows. Let G denote an arbitrary group of finite virtual cohomological dimension $\mathrm{vcd}(G)$. Put

$$d(G)=\gcd\{[G:\Lambda] \text{ where } \Lambda\subset G \text{ runs over all torsion-free subgroups of finite index}\}.$$

By a transfer argument (cf. [1]) one infers

$$d(G)H^k(G;\mathbb{Z})=0 \quad \text{if } k>\mathrm{vcd}(G).$$

Since there is a map $\theta:\Gamma_g\to \mathrm{Gl}_{2g}(\mathbb{Z})$ with torsion-free kernel, every torsion-free subgroup $\Lambda\subset \mathrm{Gl}_{2g}(\mathbb{Z})$ of finite index gives rise to a torsion-free subgroup $\theta^{-1}(\Lambda)\subset\Gamma_g$, and $[\Gamma_g:\theta^{-1}(\Lambda)]$ divides $[\mathrm{Gl}_{2g}(\mathbb{Z}):\Lambda]$. Hence

$$d(\Gamma_g)\,|\,d(\mathrm{Gl}_{2g}(\mathbb{Z})).$$

It is a classical result of Minkowski's (cf. [11] and [12]) that

$$d(\mathrm{Gl}_{2g}(\mathbb{Z}))\,\Big|\,\prod_{j=1}^{g} 4E_{2j}$$

where $E_{2j}=\mathrm{den.}(B_{2j}/2j)$ as in the Introduction. We have thus proved the following.

Proposition 3.6. *Let $x\in H^k(\Gamma_g;\mathbb{Z})$ with $k>\mathrm{vcd}(\Gamma_g)$. Then*

$$\left(\prod_{j=1}^{g} 4E_{2j}\right)x=0.$$

Although this bound is in general very crude, it is interesting to notice that it involves the same numbers (the E_{2j}'s) which actually occur as orders of torsion classes in the cohomology of the mapping class groups (cf. Section 4).

We wil now construct in an explicit way torsion in $H^*(\Gamma_g;\mathbb{Z})$ which comes from torsion in Γ_g. Let

$$\theta:\Gamma_g\to\mathrm{Gl}_{2g}(\mathbb{Z})$$

be the representation given by the action of Γ_g on $H_1(S_g;\mathbb{Z})\cong\mathbb{Z}^{2g}$. If $\mathbb{Z}/n\mathbb{Z}\subset\Gamma_g$, we shall write

$$\theta_n:\mathbb{Z}/n\mathbb{Z}\to\mathrm{Gl}_{2g}(\mathbb{Q}) \tag{3.7}$$

for θ considered as a $\mathbb{Q}$-representation and restricted to $\mathbb{Z}/n\mathbb{Z}$. Recall that the Chern classes $c_i(\theta)\in H^{2i}(\Gamma_g;\mathbb{Z})$ are of finite order dividing $2E_i$, where for i even E_i equals the denominator of B_i/i, B_i the i-th Bernoulli number (we use the convention $B_2=1/6$, $B_4=1/30,\dots$). If one is able to determine the order of the Chern class $c_i(\theta_n)\in H^{2i}(\mathbb{Z}/n\mathbb{Z};\mathbb{Z})$, then one obtains a lower bound for the order of the torsion class $c_i(\theta)\in H^{2i}(\Gamma_g;\mathbb{Z})$.

The character χ_n of θ_n can easily be computed from the Lefschetz-Hopf trace formula. If $x\in\mathbb{Z}/n\mathbb{Z}\subset\Gamma_g$ is represented by an orientation preserving periodic diffeomorphism $x:S_g\to S_g$, the alternating sum of traces

$$\sum(-1)^i\mathrm{tr}(x_*:H_iS_g\to H_iS_g)=2-\mathrm{tr}(x_*:H_1S_g\to H_1S_g)$$

equals $\mathrm{FP}(x)$, the number of fixed points of $x:S_g\to S_g$, if we assume $x\neq\mathrm{Id}$. The character χ_n of θ_n is therefore given by

$$\chi_n(x)=2-\mathrm{FP}(x),\quad x\in\mathbb{Z}/n\mathbb{Z},\ x\neq\mathrm{Id}. \tag{3.8}$$

We can now prove the following

Theorem 3.9. *Let $p^\alpha > 2$ be a prime power. Then the Chern class*

$$c_{\phi(p^\alpha)}(\theta) \in H^{2\phi(p^\alpha)}(\Gamma_{g(p^\alpha)}; \mathbb{Z})$$

is a torsion class such that all powers $(c_{\phi(p^\alpha)}(\theta))^j$, $j \geq 1$, have order a multiple of p^α.

Proof. As argued in the proof of (3.2), there is a subgroup $\mathbb{Z}/p^\alpha\mathbb{Z} \subset \Gamma_{g(p^\alpha)}$, $g(p^\alpha) = \phi(p^\alpha)/2$, such that the associated branched covering has three branch points $y_1, y_2, y_3 \in S^2$ of orders p^α, p^α and p respectively. We will show that the $\mathbb{Q}$-representation $\theta_{p^\alpha} : \mathbb{Z}/p^\alpha\mathbb{Z} \to \mathrm{Gl}_{\phi(p^\alpha)}(\mathbb{Q})$ associated with $\mathbb{Z}/p^\alpha\mathbb{Z} \subset \Gamma_{g(p^\alpha)}$ is equivalent to the cyclotomic representation σ_{p^α} considered in Section 1. Let $x \in \mathbb{Z}/p^\alpha\mathbb{Z} \subset \Gamma_{g(p^\alpha)}$ with $x \neq \mathrm{Id}$ and write also $x : S_{g(p^\alpha)} \to S_{g(p^\alpha)}$ for the corresponding covering transformation. If $x^p \neq \mathrm{Id}$ (thus $\alpha > 1$) we have $\mathrm{FP}(x) = 2$, since $\pi^{-1}(y_1)$ and $\pi^{-1}(y_2)$ will be the only fixed points of x. Hence, by (3.8),

$$\chi_{p^\alpha}(x) = 0 \quad \text{if } x^p \neq \mathrm{Id}$$

where χ_{p^α} denotes the character of θ_{p^α} as before. if $x^p = \mathrm{Id}$ but $x \neq \mathrm{Id}$, then x fixes the set $\{\pi^{-1}(y_1), \pi^{-1}(y_2), \pi^{-1}(y_3)\} \subset S_{g(p^\alpha)}$, which consists of $2 + p^{\alpha-1}$ points. Thus

$$\chi_{p^\alpha}(x) = -p^{\alpha-1} \quad \text{if } x^p = \mathrm{Id} \text{ but } x \neq \mathrm{Id}.$$

The degree of θ_{p^α} is $2g(p^\alpha) = \phi(p^\alpha)$ and therefore the $\mathbb{Q}$-representations θ_{p^α} and σ_{p^α} are equivalent, since they have the same characters (for the character of σ_{p^α} see (1.1)). The cohomological restriction of $c_{\phi(p^\alpha)}(\theta)$ to $H^{2\phi(p^\alpha)}(\mathbb{Z}/p^\alpha\mathbb{Z}; \mathbb{Z})$, induced by the inclusion $\mathbb{Z}/p^\alpha\mathbb{Z} \subset \Gamma_{g(p^\alpha)}$, is $c_{\phi(p^\alpha)}(\theta_{p^\alpha}) = c_{\phi(p^\alpha)}(\sigma_{p^\alpha})$ and the theorem follows therefore from (1.2).

Remark 3.10. We note for later use that for $p = 2$ the construction in the proof of (3.9) provides for $n = 2^\beta > 2$ a branched covering

$$S_{2^{\beta-2}} \to S^2$$

with group $\mathbb{Z}/2^\beta$ and branch point of orders $2^\beta, 2^\beta, 2$ such that the associated representation

$$\theta_{2^\beta} : \mathbb{Z}/2^\beta\mathbb{Z} \to \mathrm{Gl}_{2^{\beta-1}}(\mathbb{Q})$$

is equivalent to the cyclotomic representation σ_{2^β}.

4. Stable torsion

The goal of this section is to prove the Main Theorem stated in the Introduction. This is done by exhibiting suitable subgroups $\mathbb{Z}/n\mathbb{Z} \subset \Gamma_g$, which will permit us to analyse the Chern classes of the natural representation

$$\theta : \Gamma_g \to \mathrm{Gl}_{2g}(\mathbb{Z}).$$

Theorem 4.1. *Suppose $g > (2n)^2$. Then the following holds.*

(i) *Γ_g contains an element of order n.*

(ii) *If p is an odd prime and if $\mathbb{Z}/p^\alpha\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z}$ denotes the p-torsion subgroup, then there exists an embedding $\mathbb{Z}/n\mathbb{Z} \subset \Gamma_g$ such that the $\mathbb{Q}$-representation θ_{p^α} associated with $\mathbb{Z}/p^\alpha\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z} \subset \Gamma_g$ involves the cyclotomic representation σ_{p^α} with multiplicity prime to p.*

(iii) *Suppose $n = 2^\beta$ with $\beta \geq 2$. Then there exists an embedding $\mathbb{Z}/2^\beta\mathbb{Z} \subset \Gamma_g$ such that θ_{2^β} involves σ_{2^β} with odd multiplicity.*

Proof. We start by writing $2g$ in the form $2cn - 2d$ with $0 \leq d < n$. Thus $c > 2n$ and $2g = 2(c-d)n + 2d(n-1)$. From (2.2) we infer that there is a branched covering $S_g \to S_{(c-d)}$ with $2d+2$ branch points, each one of order n, and covering transformation group $\mathbb{Z}/n\mathbb{Z}$. Since every non-trivial covering transformation on S_g has precisely $2d+2$ fixed points, the Lefschetz–Hopf trace formula (3.8) implies that the action of the covering transformation group on $H_1(S_g) \cong \mathbb{Z}^{2g}$ has character χ given by

$$\chi(T) = -2d, \quad \text{if } T \neq \mathrm{Id}.$$

On the other hand, the characters χ_{reg} (χ_{red}) of the regular (reduced regular) representation of $\mathbb{Z}/n\mathbb{Z}$ are

$$\chi_{\mathrm{reg}}(x) = 0 \quad \text{if } x \neq \mathrm{Id}, \qquad \chi_{\mathrm{reg}}(\mathrm{Id}) = n,$$

$$\chi_{\mathrm{red}}(x) = -1 \quad \text{if } x \neq \mathrm{Id}, \qquad \chi_{\mathrm{red}}(\mathrm{Id}) = n-1.$$

Hence $\chi = 2(c-d)\chi_{\mathrm{reg}} + 2d\chi_{\mathrm{red}}$. In particular, χ is faithful and the class of a generator of the covering transformation group represents an element of order n in Γ_g (this follows too from 2.3); we write $\mathbb{Z}/n \subset \Gamma_g$ for the corresponding subgroup. Let p be any prime and $\mathbb{Z}/p^\alpha \subset \mathbb{Z}/n \subset \Gamma_g$ the p-torsion subgroup of $\mathbb{Z}/n\mathbb{Z}$. As before, we write θ_{p^α} for the associated representation $\mathbb{Z}/p^\alpha\mathbb{Z} \to \mathrm{Gl}_{2g}(\mathbb{Q})$. The cyclotomic representation σ_{p^α} occurs in the restriction of the regular (reduced regular) $\mathbb{Q}$-representation of $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/p^\alpha\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z}$ with multiplicity n/p^α, which is prime to p. Thus, the multiplicity of σ_{p^α} in θ_{p^α} is $(2(c-d)+2d)n/p^\alpha = 2cq$, with q prime to p. We consider now the case p odd. If c is already prime to p the multiplicity of σ_{p^α} in θ_{p^α} is prime to p and we are done. If p divides c, we rewrite $2g = 2(c-d)n + 2d(n-1)$ in the form $2g = 2((c-d)-(n-1))n + (2d+2n)(n-1)$, noting that $c-d-(n-1) > 0$ since $c > 2n$ and $0 \leq d < n$. As before, we obtain $\mathbb{Z}/p^\alpha\mathbb{Z} \subset \mathbb{Z}/n\mathbb{Z} \subset \Gamma_g$ associated to the n-fold branched covering $S_g \to S_{(c-d-(n-1))}$, and the multiplicity of σ_{p^α} in θ_{p^α} is now

$$(2((c-d)-(n-1)) + (2d+2n))n/p^\alpha = (2c+2)q,$$

which is prime to p.

For (iii) we put $n=2^\beta$, $\beta\geq 2$. Let $h=g-2^{\beta-2}>2^{2\beta+1}$. We can write $h=2(c-d)2^\beta+2d(2^\beta-1)$ with $c-d>0$ and $0\leq d<2^\beta$. Hence, as it the case p odd, we obtain a branched covering $S_h\to S_{(c-d)}$ with group $\mathbb{Z}/2^\beta\mathbb{Z}\subset\Gamma_h$ for which σ_{2^β} occurs with even multiplicity (of the form $2cq$) in θ_{2^β}. On the other hand, there is, according to (3.10), a branched covering $S_{2^{\beta-2}}\to S^2$ with group $\mathbb{Z}/2^\beta\mathbb{Z}$ for which the induced representation on $H_1(S_{2^{\beta-2}};\mathbb{Q})$ is equivalent to σ_{2^β}. Choosing a fixed point in S_h and $S_{2^{\beta-2}}$ for the $\mathbb{Z}/2^\beta\mathbb{Z}$-action, we can form the equivariant connected sum $S_g=S_h\# S_{2^{\beta-2}}$, which carries a $\mathbb{Z}/2^\beta\mathbb{Z}$ action such that σ_{2^β} has odd multiplicity in θ_{2^β}, as we see from the $\mathbb{Z}/2^\beta\mathbb{Z}$-equivariant decomposition

$$H_1(S_g)\cong H_1(S_h)\oplus H_1(S_{2^{\beta-2}}).$$

Therefore, the so constructed subgroup $\mathbb{Z}/2^\beta\mathbb{Z}\subset\Gamma_g$ has the desired property.

Corollary 4.2. *Let p be a prime and suppose $g>(2p^\alpha)^2$.*

(i) *If p is odd, then for every $k\geq 1$*

$$H^{2k\phi(p^\alpha)}(\Gamma_g;\mathbb{Z})$$

contains an element of order p^α.

(ii) *If $p=2$ and $\alpha>1$, then for every $k\geq 1$*

$$H^{2k\phi(2^\alpha)}(\Gamma_g;\mathbb{Z})=H^{2^\alpha k}(\Gamma_g;\mathbb{Z})$$

contains an element of order 2^α.

Proof. This is an immediate consequence of (1.2) in conjunction with (4.1) (ii) and (iii) respectively.

Remark 4.3. Statement (ii) of (4.2) is false for $\alpha=1=k: H^2(\Gamma_g;\mathbb{Z})$ is torsion-free for $g\geq 3$ (cf. [13]).

We can now prove the following theorem.

Theorem 4.4. *Suppose $g>(8m)^2$. Then $H^{4m}(\Gamma_g;\mathbb{Z})$ contains an element of order $E_{2m}=\text{den.}(B_{2m}/2m)$.*

Proof. It is well known that p^β divides E_{2m} if and only if $2m\equiv 0 \bmod \phi(p^\beta)$ (cf. [3]). Suppose now that p^α denotes the highest power of p dividing E_{2m}; then $4m=2k\phi(p^\alpha)=2kp^{\alpha-1}(p-1)$ for some k, and therefore $4m\geq p^\alpha$, which implies $(8m)^2\geq(2p^\alpha)^2$. Since we assume $g>(8m)^2$, we conclude by (4.2) that $H^{4m}(\Gamma_g;\mathbb{Z})$ contains an element of order p^α. Since p was an arbitrary prime, it follows that $H^{4m}(\Gamma_g;\mathbb{Z})$ contains an element of order E_{2m}.

According to [9], $H^j(\Gamma_g;\mathbb{Z})$ is independent of g for $g>3j$. By choosing g sufficiently large we see that (4.4) implies that $H^{4m}(\Gamma)$ always contains an element of order E_{2m}, which completes the proof of the Main Theorem.

As a consequence of the Main Theorem and the discussion of the torsion gaps for the Γ_g's in Section 3, it is now possible to produce examples such that $H^*(\Gamma_g; \mathbb{Z})$ has p-torsion but Γ_g does not.

Corollary 4.5. *Let $p \geq 17$ be a prime and let $\kappa(p) = p(p-3)/2 - (p-1)/2$. Then $\Gamma_{\kappa(p)}$ has no p-torsion but $H^{2(p-1)}(\Gamma_{\kappa(p)}; \mathbb{Z})$ contains an element of order p.*

Proof. According to (3.4), $\Gamma_{\kappa(p)}$ has no p-torsion. Since $3 \cdot 2(p-1) < \kappa(p)$, we infer $H^{2(p-1)}(\Gamma_{\kappa(p)}; \mathbb{Z}) = H^{2(p-1)}(\Gamma)$ and since p divides E_{p-1}, we conclude that $H^{2(p-1)}(\Gamma_{\kappa(p)}; \mathbb{Z})$ contains an element of order p.

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